BAYESIAN AND CREDIBILITY ESTIMATION FOR THE CHAIN LADDER RESERVING METHOD

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Abstract: Gisler and Wuthrich [7] describe how to get reserve estimates by means of Credibility and the Bayesian estimators based on the development factors from different lines of business. This approach allows to combine individual and collective claims information to get the best estimation of the unknown reserves.
The purpose of this paper is to compare the reserves estimates and the mean square error of prediction between the Credibility and Bayesian models. Both models include the link to the classical chain ladder model for comparison purposes. Finally, we include a way of implement the Bayesian model using Markov Chain Monte Carlo methods with the programming tool WinBUGS [13].

Key Words: Bayesian Models, Chain-Ladder, Credibility Theory, Markov Chain Monte Carlo, Normal Family.

1- Introduction

The determination of claim reserves for the outstanding liabilities is one of the most important tasks that an actuary performs to preserve the financial solvency of an insurance company.

The usual way to obtain estimates about the unknown claim amounts for future years has been the use of forecasting methods based on the historical information.

In some cases, the lack of information about past claims can constitute an obstacle to the determination of reliable reserves. For that reason, actuaries often consider on the one hand the market experience and on the other one the company's own experience: collective and individual information in credibility terminology. In this way, it is possible to add more information about the corresponding line of business.

Thus, credibility models allow the individual experience to be combined with the collective by introducing an optimal factor which measures the weight between the individual and collective claims information.

Bayesian models also include the prior information (collective) in a natural way. However, they allow more statistical information about the reserves estimates to be included, and also enable us to obtain the complete predictive distribution of the possible outcomes, in order to study risk measures.

In this paper, we focus on Bayesian models to estimate the claim reserving amounts using the Markov chain Monte Carlo (MCMC) methods. In particular, we investigate the difference between credibility and Bayesian approaches to statistical reasoning and model estimation. We focus on the chain-ladder technique, and thus also include models which are similar to those of Mack [10]. The code in which the Bayesian MCMC model can be implemented by the statistical package WinBUGS [13] is also included.

This paper is structured as follows. In the second section we summarize the traditional chain ladder method (CLM). The third section states the modeling assumptions of Mack [10] and introduces the way in which the variability of the reserve estimates can be evaluate through the mean square error (MSE) of prediction. Credibility theory is the topic of the fourth section. The fifth describes the Bayesian formulation, including the implementation using WinBUGS [13]. Section 6 is devoted to the conclusions about the comparison of the results from the models.

2- The Chain-Ladder Method

In the run-off triangle, each row represents an origin year i for $0 \leq i \leq I$ and the column represents the development year, j, for $0 \leq j \leq J$. $C_{i,j}$ denotes cumulative claims (either incurred or paid) with a delay of j years from the origin year i.

Usually, the data consist of a triangle where $I = J$. However, other shapes of claim data can be assumed. In particular, we assume that the data information have an irregular pentagon shape where $I > J$ as in Table (1). Thus, the data consist of known cumulative claims for $i + j \leq I$ and unknown cumulative claims for $i + j > I$. In this paper we define $\boldsymbol{C}$ as the complete observed information for $i + j \leq I$ to simplify the notation.

The column sum of the observed cumulative claims is defined as

$$S_j^k = \sum_{i=0}^{k} C_{i,j} \tag{2.1}$$

Using this notation, the standard chain-ladder, development factors can be calculated as

$$f_j = \sum_{i=0}^{I-j-1} C_{i,j+1} \Big/ \sum_{i=0}^{I-j-1} C_{i,j} = S_{j+1}^{I-j-1} \big/ S_j^{I-j-1} \text{, for } 0 \leq j \leq J-1 \tag{2.2}$$

Table 1. Loss Development Data Structure

Origin Year	i / j	Development Year 0	1	…	j	…	J − 1	J
	0	$C_{0,0}$	$C_{0,1}$	…	$C_{0,j}$	…	$C_{0,J-1}$	$C_{0,J}$
	1	$C_{1,0}$	$C_{1,1}$	…	$C_{1,j}$	…	$C_{1,J-1}$	$C_{1,J}$
	⋮	⋮	⋮	…	⋮	…	⋮	⋮
	i = J	$C_{J,0}$	$C_{J,1}$	…	$C_{J,j}$	…	$C_{J,J-1}$	$C_{J,J}$
	i = J+1	$C_{J+1,0}$	$C_{J+1,1}$	…	$C_{J+1,j}$	…	$C_{J+1,J-1}$	
	⋮	⋮	⋮	⋮	⋮			
	I - 2	$C_{I-2,0}$	$C_{I-2,1}$	$C_{I-2,2}$				
	I - 1	$C_{I-1,0}$	$C_{I-1,1}$					
	I	$C_{I,0}$						

Appendix (A) shows the data (claim amounts) from different lines of business used on this article, they have been taken from Gisler and Wuthrich [7].

The aim of the CLM is to complete the empty triangle on the lower right corner of the table with the help of the development factors. In this paper it is assumed that the claim amount for the rows $i \leq J$ has fully development and therefore we apply the development factors to the latest amounts known for the rest of the rows $(i > J)$ to estimate of the unknown claim amounts:

$$\hat{C}_{i,j}^{CLM} = C_{i,I-i} * \prod_{j=I-i}^{J-1} f_j, \text{ for } i > J, j > I-i \quad \textbf{(2.3)}$$

In this way, it is possible to estimate the ultimate cumulative $C_{i,J}^{CLM}$ and obtain the reserve estimate for each accident year i:

$$R_i^{CLM} = \hat{C}_{i,J}^{CLM} - C_{i,I-i}, \text{ for } i > J \quad \textbf{(2.4)}$$

Finally, we can find the estimate of the total amount of outstanding claims as

$$R_{Total}^{CLM} = \sum_{i=J+1}^{I} \hat{C}_{i,J} - \sum_{i=J+1}^{I} C_{i,I-i}, \text{ for } i > J \quad \textbf{(2.5)}$$

3- Mack's Model

Mack [9] investigated the CLM, assuming a distribution-free model and specifying the first two moments for the cumulative claims, based on the following weak assumptions:

A1) Independence for the random variables $C_{i,j}$ *between different accident years* i.

A2) Existence of unknown factors $f_j > 0$ and $\sigma_j^2 > 0$, such that

$$E\left[C_{i,j+1} \middle| C_{i,0}, \dots, C_{i,j}\right] = f_j C_{i,j}, \text{ for } 0 \le i \le I, 0 \le j \le J-1 \tag{3.1}$$

$$Var\left[C_{i,j+1} \middle| C_{i,0}, \dots, C_{i,j}\right] = \sigma_j^2 C_{i,j}, \text{ for } 0 \le i \le I, 0 \le j \le J-1 \tag{3.2}$$

If we define the individual development factors by

$$\lambda_{i,j} = \frac{C_{i,j+1}}{C_{i,j}}, \text{ for } i + j < I \tag{3.3}$$

The assumptions (3.1) and (3.2) can be rewritten into the form

$$E\left[\lambda_{i,j} \middle| C_{i,0}, \dots, C_{i,j}\right] = f_j, \text{ for } 0 \le i \le I-1, 0 \le j \le J-1 \tag{3.4}$$

$$Var\left[\lambda_{i,j} \middle| C_{i,0}, \dots, C_{i,j}\right] = \frac{\sigma_j^2}{C_{i,j}}, \text{ for } 0 \le i \le I-1, 0 \le j \le J-1 \tag{3.5}$$

These assumptions allow the variability of the estimates of the development factors to be quantified. Finally, Mack [10] includes a formula for the MSE of prediction, which measures the average distance between the forecast $\hat{C}_{i,J}$ and the future realization $C_{i,J}$ as

$$MSE\left(R_i^{MCL}\right) = E\left[\left(\hat{C}_{i,J} - C_{i,J}\right)^2 \middle| \mathbf{C}\right] = Var\left(C_{i,J} \middle| \mathbf{C}\right) + \left(E\left(C_{i,J} \middle| \mathbf{C}\right) - \hat{C}_{i,J}\right)^2 \tag{3.6}$$

For aggregation of all years, the MSE is defined as

$$MSE\left(R_{Total}^{MCL}\right) = E\left[\left(\sum_{i=J+1}^{I} \hat{C}_{i,J} - \sum_{i=J+1}^{I} C_{i,J}\right)^2 \middle| \mathbf{C}\right] \tag{3.7}$$

where the summation starts in the year $i = J+1$ because the claim amouts are known for $i \le J$.

Other stochastic models for the CLM have also been investigated in order to calculate measures of uncertainty, and also the predictive distribution of outstanding claims. For a review of these stochastic extensions see England and Verrall [4], Wuthrich and Merz [15] and Hess and Schmidt [8].

In this paper we compare the Bayesian reserve estimates with the Credibility estimations based in the Mack model. We will see that under certain conditions and by using non-informative priors, the reserve forecast for both Bayesian and Credibility model are similar to the CLM. However the estimator of the MSE is different.

4- Credibility Theory for the Chain Ladder Model

Credibility theory obtains the best estimate for the development factors based on the information for the triangle under consideration (the individual line of business) and collective claims information from similar lines of business. To allow us to include different lines of business, a third index is added to the notation, k, for $0 \le k \le K$. In this way we can distinguish different loss development data structure as Appendix (A).

Gisler and Wuthrich [7] formulated the credibility estimators for the chain ladder factors f_j by

$$f_{j,k}^{Cred} = \alpha_{j,k} f_{j,k}^{Ind} + \left(1 - \alpha_{j,k}\right) f_j^{Coll} \tag{4.1}$$

where

- $f_{j,k}^{Cred}$ is a weighted mean from the individual and collective development factors.
- $f_{j,k}^{Ind}$ is the individual development factor for each line of business k.

$$f_{j,k}^{Ind} = S_{j+1,k}^{I-j-1} / S_{j,k}^{I-j-1} = E\left[\lambda_{i,j,k} \left| C_{i,0,k}, \dots, C_{i,j,k}\right.\right], \text{ for } 0 \le j \le J-1, 0 \le k \le K \tag{4.2}$$

- f_j^{Coll} is the collective development factor for all the lines of business (prior knowledge).

$$f_j^{Coll} = \sum_{k=1}^{K} \left(S_{j+1,k}^{I-j-1} \middle/ S_{j,k}^{I-j-1} \right) = E\left(\mu\left(f_{j,k}\right)\right), \text{ for } 0 \le j \le J-1 \tag{4.3}$$

- $\alpha_{j,k}$ is a parameter used to weight the individual and collective development factors.

$$\alpha_{j,k} = \frac{S_{j,k}^{I-j-1}}{S_{j,k}^{[I-j-1]} + \frac{\sigma_{j,k}^2}{\tau_j^2}}, \text{ for } 0 \le j \le J-1 \tag{4.4}$$

- $\sigma_{j,k}^2$ is the variance for the individual development factors.

$$\sigma_{j,k}^2 = \frac{1}{I-j-1} \sum_{i=0}^{I-j} C_{i,j,k} \left(\lambda_{i,j,k} - f_{j,k}^{Ind}\right)^2, \text{ for } 0 \le j \le J-2 \tag{4.5}$$

- τ_j^2 is the variance of the collective factors, estimated as in Buhlmann and Gisler [3], Chapter 4.

Thus, the unknown claim amounts $C_{i,J,k}$ for the rows $(i > J)$ are estimated using the development factors $f_{j,k}^{Cred}$

$$\hat{C}_{i,j,k}^{Cred} = C_{i,I-i,k} * \prod_{j=I-i}^{J-1} f_{j,k}^{Cred}, \text{ for } i > J,\ j > I-i,\ 0 \le k \le K \qquad \textbf{(4.6)}$$

In this way, it is possible to obtain the reserve estimate for each year i

$$R_{i,k}^{Cred} = \hat{C}_{i,J,k}^{Cred} - C_{i,I-i,k}, \text{ for } i > J \qquad \textbf{(4.7)}$$

and the corresponding, total reserve

$$R_{Total,k}^{Cred} = \sum_{i=J+1}^{I} \hat{C}_{i,J,k}^{Cred} - \sum_{i=J+1}^{I} C_{i,I-i,k}, \text{ for } i > J \qquad \textbf{(4.8)}$$

Observe that this approach only uses the two first moments assumptions (3.4) and (3.5) as in Mack´s model. However, the reserve distribution is not available except when approximation assumptions are made.

The formula measures the variability (MSE) of the reserves estimations of prediction $R_{i,k}^{Cred}$ is the same as formula (3.6).

Tables (2) and (3), summarize the estimated values for the credibility method.

Table 2. Estimates of individual $\sigma_{j,k}^2$

k / j	0	1	2	3	4	5	6	7	8	9
0	418.84	87.39	6.98	1.53	1.02	7.07	18.99	0.66	0.54	0.00
1	176.15	11.25	2.65	0.38	0.71	0.00	2.66	0.00	0.00	0.87
2	58.60	6.56	9.48	28.07	0.04	0.05	0.32	0.05	0.00	0.00
3	317.92	38.22	12.97	0.61	0.72	17.28	1.43	0.56	0.00	0.00
4	134.69	14.64	6.34	4.98	0.06	0.40	2.05	0.16	0.05	0.00
5	912.98	50.36	8.73	0.03	0.00	1.25	0.03	0.00	0.00	0.06

Table 3. Development factors

j	0	1	2	3	4	5	6	7	8	9
$f_{j,0}^{Ind}$	2.270	1.233	0.982	1.024	1.012	0.981	0.962	1.003	0.996	1.000
$f_{j,0}^{Cred}$	2.111	1.189	0.996	1.015	1.004	1.001	0.984	0.998	1.000	0.999
$f_{j,1}^{Ind}$	2.133	1.094	1.032	1.002	0.998	1.000	1.014	0.999	1.000	0.990
$f_{j,1}^{Cred}$	2.111	1.111	1.033	1.012	1.003	1.001	0.997	0.998	1.000	0.997
$f_{j,2}^{Ind}$	2.189	1.138	1.037	1.042	1.003	1.000	0.999	1.002	1.000	1.000
$f_{j,2}^{Cred}$	2.111	1.134	1.036	1.016	1.004	1.001	0.994	0.998	1.000	0.999
$f_{j,3}^{Ind}$	2.108	1.070	1.054	1.013	1.004	1.015	0.996	0.995	1.000	1.000
$f_{j,3}^{Cred}$	2.111	1.084	1.050	1.013	1.004	1.001	0.994	0.998	1.000	0.999
$f_{j,4}^{Ind}$	1.930	1.114	1.018	0.995	1.002	0.997	0.999	0.997	1.002	1.000
$f_{j,4}^{Cred}$	2.111	1.119	1.021	1.010	1.004	1.001	0.995	0.998	1.000	0.999
$f_{j,5}^{Ind}$	3.008	1.190	1.146	1.006	1.000	0.979	0.996	1.000	1.000	1.004
$f_{j,5}^{Cred}$	2.111	1.139	1.064	1.013	1.004	1.001	0.993	0.998	1.000	0.999
f_{j}^{Coll}	2.111	1.129	1.033	1.013	1.004	1.001	0.993	0.998	1.000	0.999

5- Bayesian Models

Some applications of Bayesian models for outstanding reserve can be founded in Alba [1], England and Verrall [4] [5] and Ntzoufraz and Dellaportas [11]. Programing implementation of Bayesian models apply to IBNR reserves can be found in Alba [2], Scollnik [12], and Verrall [14].

This paper used the hierarchical approach for the Bayesian implementation of the model with three stages.

The first stage for the implementation of Bayesian model consists in defines a likelihood function $\pi\left(\lambda_{i,j,k}^{U} \middle| f_{j,k}^{Ind}\right)$ to describe the known development factors λ^{U} (upper left corner of the table) for $\left\{\lambda_{i,j,k}^{U} : \ j=0,\dots,J-1; i=0,\dots,I-1; k=0,\dots,K\right\}$. For that we can choose any distribution from the exponential family.

To make Bayesian model comparable to the Mack's model, we complement the assumption (3.4) and (3.5) with a Normal distribution for $\lambda_{i,j,k}^{U}$. Then, likelihood function for of the development factor λ^{U} is defined by

$$\lambda_{i,j,k}^{U} \left| f_{j,k}^{Ind}, \upsilon_{i,j,k}^{2} \right. \sim N\left(f_{j,k}^{Ind}, \upsilon_{i,j,k}^{2}\right) \quad \textbf{(5.1)}$$

where $\lambda_{i,j,k}$ is the individual development factors as (3.3) and $\upsilon_{i,j,k}^{2} = \frac{\sigma_{j,k}^{2}}{C_{i,j,k}}$ is the variance known , obtained with the individual $\sigma_{j,k}^{2}$ obtained as (4.5) in Table (2).

Observe that table (2) contains some values equal to cero. These values can't be employed in the implementation of BUGS code, therefore, we apply the next approximation

$$\sigma_{j,k}^{2} = \begin{cases} 1/1000 & for \quad \sigma_{j,k}^{2} = 0 \\ \sigma_{j,k}^{2} & for \quad \sigma_{j,k}^{2} \neq 0 \end{cases} \quad \textbf{(5.2)}$$

The second stage contain a prior about the parameter $f_{j,k}^{Ind}$ given f_{j}^{Coll}. Again we can suppose a Normal distribution

$$f_{j,k}^{Ind} \left| f_{j}^{Coll} \right. \sim dnorm\left(f_{j}^{Coll}, \tau_{j,k}^{2}\right) \quad \textbf{(5.3)}$$

From the credibility formula (4.1) we can observe that if we define large variances over $\tau_{j,k}^{2}$, the parameter $\alpha_{j,k} \to 1$. In other words, the credibility forecast coincides with the classical chain ladder forecast. Then we consider large variance over $\tau_{j,k}^{2}$ with

$$\tau_{j,k}^{2} = 1000 \quad \textbf{(5.4)}$$

This is a non informative prior density which reflects a total lack or ignorance of information.

Finally the third stage of the hierarchical model contain vague independent normal priors on θ_j used to generate the development factors f_{j}^{Coll}.

$$\theta_j \sim dnorm\left(\mu_0, \kappa^2\right), \text{ with } \mu_o = 0, \quad \kappa^2 = 1000 \quad \textbf{(5.5)}$$

$$\log\left(f_{j}^{Coll}\right) = \theta_j \quad \textbf{(5.6)}$$

Summarizing, the model is defined by the three levels as

$$\begin{aligned}
\lambda^{U}_{i,j,k} \left| f^{Ind}_{j,k}, \upsilon^{2}_{i,j,k} \right. &\sim N\left(f^{Ind}_{j,k}, \upsilon^{2}_{i,j,k}\right) \\
f^{Ind}_{j,k} \left| f^{Coll}_{j}, \tau^{2}_{j,k} \right. &\sim N\left(f^{Coll}_{j}, \tau^{2}_{j,k}\right) \\
\theta_j \left| \mu_0, \kappa \right. &\sim N\left(\mu_0, \kappa\right) \\
\log\left(f^{Coll}_{j}\right) &= \theta_j
\end{aligned} \tag{5.7}$$

Then all the existing information about the model is introduced via Bayes's Theorem by

$$\pi(f^{Coll}_{j,k}, f^{Ind}_{j} \left| \lambda^{U}_{i,j,k}\right.) \propto \pi\left(\lambda^{U}_{i,j,k} \left| f^{Ind}_{j,k}\right.\right) * \pi\left(f^{Ind}_{j,k} \left| f^{Coll}_{j}\right.\right) * \pi\left(f^{Coll}_{j}\right) \tag{5.8}$$

We can estimate the prediction of the future development factors by means of the posterior predictive distribution. This distribution considers the posterior distribution from the known development factors λ^{U} and the information about the unknown claim amounts λ^{L} (the lower right corner of the table).

$$\pi\left(\lambda^{L}_{i,j} \left| \lambda^{U}_{i,j}\right.\right) = \int \pi\left(\lambda^{L}_{i,j} \left| f^{Ind}_{j,k}, f^{Coll}_{j}\right.\right) \pi\left(f^{Ind}_{j,k}, f^{Coll}_{j} \left| \lambda^{U}_{i,j}\right.\right) \partial f^{Ind} \partial f^{Coll}, \quad \text{for } i + j > N + 1 \tag{5.9}$$

Unfortunately, the analysis for the marginal posterior distribution $f^{Ind}_{j,k}, f^{Coll}_{j} \left| \lambda^{U}_{i,j}\right.$ is not analytically tractable. However, we can obtain a numerical approximation by MCMC methods. These methods include the use of numerical integration methods as the Gibbs, which supplies samples from the conditional posterior distribution of each parameter f^{Ind} and f^{Coll}.

The first iteration of Gibbs sampling consists in random draws from the full conditional distribution of each parameter. In our case we consider the parameters f^{Ind} and f^{Coll} defined with its conditional distributions:

$$\begin{aligned}
\pi\left(f^{Ind} \left| \lambda^{U}\right.\right) &\propto \pi\left(\lambda^{U} \left| f^{Ind}\right.\right) \pi\left(f^{Ind} \left| f^{Col}\right.\right) \\
\pi\left(f^{Col} \left| f^{Ind}\right.\right) &\propto \pi\left(f^{Ind} \left| f^{Col}\right.\right) \pi\left(f^{Col}\right)
\end{aligned} \tag{5.10}$$

Appendix (B) contains the procedure to obtain the full conditional posterior for f^{Ind} and f^{Coll}. Both parameters are conditionally conjugate. More details about the properties of the conjugate families can be found in Gamerman [6]

In this way, the full conditional posterior for $f^{Ind}_{j,k} \left| \lambda^{U}_{i,j,k}\right.$ is distributed Normal with mean $f^{(1)}_{i,j,k}$ and variance $\left(\tau^{(1)}_{i,j,k}\right)^{-2}$

$$\pi\left(f_{j,k}^{Ind} \middle| \lambda_{i,j,k}^{U}\right) \quad \propto \quad \prod_{j=0}^{J-1}\left\{\exp\left\{-\frac{1}{2}\left(\frac{f_{j,k}^{Ind}-f_{i,j,k}^{(1)}}{\tau_{i,j,k}^{(1)}}\right)^2\right\}\right\} \tag{5.11}$$

And the full conditional posterior for $f_j^{Coll} \middle| f_{j,k}^{Ind}$ is distributed LogNormal with mean $f_{i,j,k}^{(2)}$ and variance $\left(\tau_{i,j,k}^{(2)}\right)^{-2}$.

$$\pi\left(f_{j,k}^{Ind} \middle| \lambda_{i,j,k}^{U}\right) \quad \propto \quad \prod_{j=0}^{J-1}\frac{1}{f_{j,k}^{Ind}}\exp\left\{-\frac{1}{2}\left[\left(\frac{f_j^{Coll}-f_{i,j,k}^{(2)}}{\tau_{i,j,k}^{(12)}}\right)^2\right]\right\} \tag{5.12}$$

Then, each parameter is updated from its conditional distribution and then we fill the first iteration with the parameters $\hat{f}_{(1)}^{Ind}$ and $\hat{f}_{(1)}^{Coll}$. To fill the next iteration, we need to update again the conditional distribution incorporating the values of the first iteration. On this way we update t times the parameters $\hat{f}_{(t)}^{Ind}$ and $\hat{f}_{(t)}^{Coll}$, in order we incorporating its last iteration $t-1$. More details about the Gibb sampling algorithm can be found in Gamerman [6].

The model implementation in the computing package WinBUGS [13] is coded in the Appendix (C).

From (5.5) and (5.12), we get the link with the credibility models by means of $f_{j,k}^{Bayes}$ as

$$f_{j,k}^{(Bayes)} = \left(\tau_{j,k}^{(2)}\right)^2\left(\tau_{j,k}^{-2}\hat{f}_{j,k}^{Ind} + \kappa^{-2}\hat{f}_j^{Coll}\right) = \alpha\hat{f}_{j,k}^{Ind} + (\alpha-1)\hat{f}_j^{Coll} \tag{5.13}$$

for $\alpha_{j,k}$ defined as (4.4).

The Bayesian MSE of prediction to measure the variability of the reserves estimations of prediction $R_{i,k}^{Bayes}$ is

$$MSE\left(R_i^{Bayes}\right) = E\left[\left(C_{i,J}-E\left(C_{i,J}\right)\right)^2 \middle| C\right] = Var\left(C_{i,J} \middle| C\right) \tag{5.14}$$

The difference between the Mack´s (3.6) and Bayesian approach (5.14) is that the first one includes two parts $Var\left(C_{i,J}|C\right)$ and $\left(E\left(C_{i,J}|C\right)-\hat{C}_{i,J}\right)^2$ to calculate the MSE, and the second ones includes all the uncertainty only on the parameter $Var\left(C_{i,J}|C\right)$, which contains the uncertain parameters from the posterior distribution of all parameters.

This is where MCMC proves to be very useful: it yields random observations for aggregate claims in each cell of the (unobserved) lower right corner of the triangle λ^L. Then, the predictive values include the MSE variability.

Subsequently, the predictive values can be used to estimate the mean and variance of the individual development factors f^{Bayes}. On this way, is possible to obtain directly the reserve with the predictive distribution of the outstanding claims.

Thus, the estimation of the unknown claim amounts $C_{i,J,k}$ for the rows $(i > J)$ is estimated by the development factors $f_{j,k}^{Bayes}$

$$\hat{C}_{i,j,k}^{Bayes} = C_{i,I-i,k} * \prod_{j=I-i}^{J-1} f_{j,k}^{Bayes}, \text{ for } i > J,\ j > I-i,\ 0 \le k \le K \qquad \textbf{(5.18)}$$

In this way, it is possible to obtain the reserve estimate for each year i

$$R_{i,k}^{Bayes} = \hat{C}_{i,J,k}^{Bayes} - C_{i,I-i,k}, \text{ for } i > J \qquad \textbf{(5.19)}$$

And the corresponding, total reserve

$$R_{Total,k}^{Bayes} = \sum_{i=J+1}^{I} \hat{C}_{i,J,k}^{Bayes} - \sum_{i=J+1}^{I} C_{i,I-i,k}, \text{ for } i > J \qquad \textbf{(5.20)}$$

Table (4) and (5) shows the development factor estimates for $\hat{f}^{Bayes}$ and the reserves R^{Bayes} with the prediction errors, respectively. An initial burn-in sample of 10,000 iterations was used. The results of these observations were discarded, to remove any effect from the initial conditions and allow the simulations to converge. Then a further 50,000 simulations for each distributional assumption were run to get the results.

Table 4. Development factors for f^{Bayes}

j/k	0	1	2	3	4	5	6	7	8	9
$f_{j,0}^{Bayes}$	2.268	1.233	0.982	1.025	1.011	0.981	0.963	1.003	0.996	1.000
$f_{j,1}^{Bayes}$	2.134	1.094	1.032	1.002	0.998	1.000	1.014	0.999	1.000	0.990
$f_{j,2}^{Bayes}$	2.189	1.138	1.037	1.042	1.003	1.000	0.999	1.002	1.000	1.000
$f_{j,3}^{Bayes}$	2.108	1.070	1.054	1.013	1.004	1.014	0.996	0.995	1.000	1.000
$f_{j,4}^{Bayes}$	1.931	1.114	1.018	0.995	1.002	0.997	0.999	0.997	1.002	1.000
$f_{j,5}^{Bayes}$	2.997	1.191	1.147	1.006	1.000	0.979	0.996	1.000	1.000	1.004

Table 5. Reserves and prediction errors for the Bayesian model

i	*k = 0*		*k = 1*		*k = 2*		*k = 3*		*k = 4*		*k = 5*	
	$R_{i,0}^{Bayes}$	*s.d*	$R_{i,1}^{Bayes}$	*s.d*	$R_{i,2}^{Bayes}$	*s.d*	$R_{i,3}^{Bayes}$	*s.d*	$R_{i,4}^{Bayes}$	*s.d*	$R_{i,5}^{Bayes}$	*s.d*
11	0.00	1.01	-8.98	29.54	0.00	1.25	0.00	2.95	0.00	1.50	0.62	3.00
12	-6.70	29.51	-10.52	31.76	0.01	1.93	-0.01	2.00	4.08	12.49	0.44	2.55
13	-5.36	68.41	-12.55	34.10	4.36	11.97	-34.32	70.02	-3.29	25.67	0.79	3.43
14	-25.05	116.00	3.31	59.46	0.90	20.67	-33.62	89.56	-3.49	63.11	0.00	4.28
15	-32.60	125.10	6.56	87.45	1.29	27.02	14.32	250.10	-4.83	53.29	-7.87	23.60
16	-24.80	123.10	1.37	74.11	4.11	24.41	28.59	270.80	-3.01	56.66	-5.10	18.87
17	-8.42	107.30	1.98	57.53	43.08	168.80	24.43	160.90	-14.08	119.40	-2.64	16.07
18	-16.55	120.20	14.20	56.23	72.32	189.80	138.50	256.40	8.48	113.30	18.12	38.17
19	76.48	256.30	54.87	92.91	144.10	185.10	225.30	352.80	155.50	199.40	16.56	692.70
20	528.80	645.30	187.10	399.20	431.90	233.40	652.00	559.80	354.80	286.30	8.49	1347.0
Σ	**485.80**	**758.80**	**237.40**	**454.60**	**702.10**	**409.20**	**1015.0**	**865.40**	**494.20**	**412.00**	**29.40**	**1513.0**

6- Conclusions

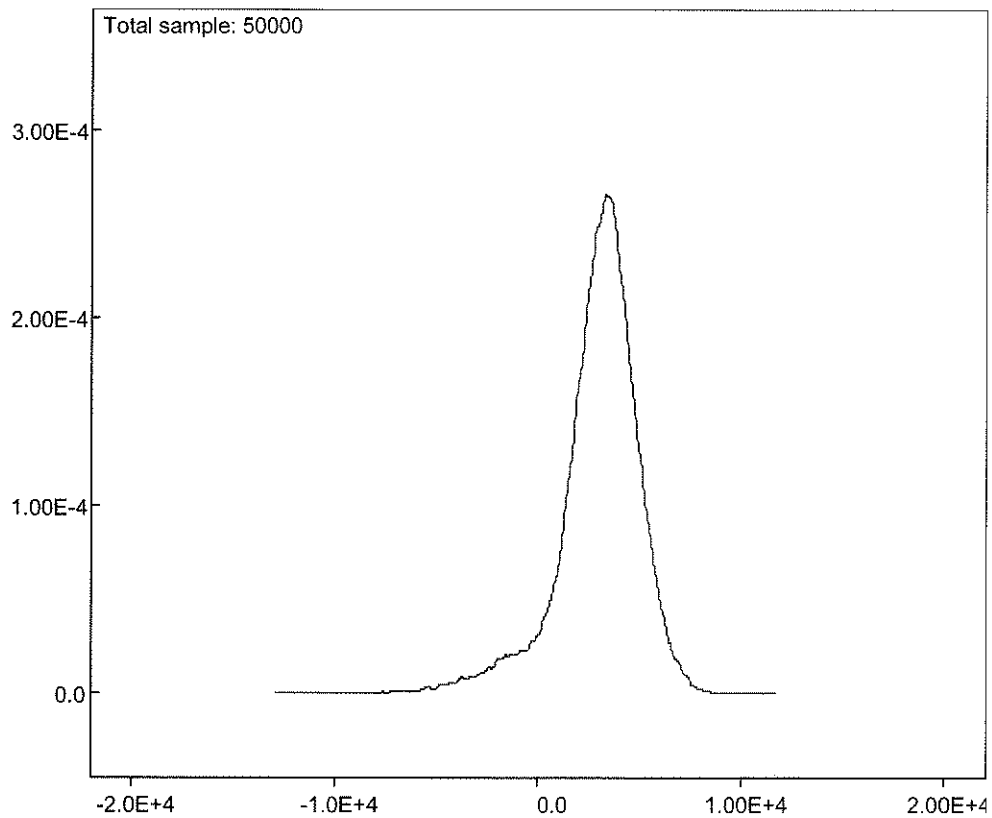
An important aspect in the constitution of loss reserving reserves is the fact that we can introduce more information with the information of different lines of business. In this way, we can develop a credibility formula which considers reserves estimates as the CLM when $\alpha = 1$ and a Credibility in other case. Moreover, the advantage of Bayesian approach is that we can obtain a full predictive distribution, rather than just the first and second moments as in Credibility and CL method. Plot (1) shows the posterior distribution for the Bayesian model.

Table (6) shows on the one hand the reserves estimates when $\alpha = 1$ (non-informative prior); on the other, the MSE of prediction for each method. The results shown, how the use of non-informative priors in Bayesian analysis leads close reserves estimates as the MLE, when fitting the same model structure over the mean. For our example the Mack´s model has the smallest error predictor. However, this model isn´t express the idea of combine different lines of business as the Credibility and Bayesian models. For both models the MSE of prediction are similar. So both models are good to adjust the claim amounts.

Table 6. Reserves for Credibility, Chain-ladder and Bayesian model

k	Reserves			MSE (overall)			
	Cred	MCL	Bayes	Cred	MCL		Bayes
					$\alpha=1$	Mack	
0	504	486	486	498	510	510	759
1	244	235	237	402	425	424	455
2	517	701	702	520	566	565	409
3	899	1029	1,015	729	765	765	865
4	621	495	494	584	593	593	412
5	25	40	29	143	163	163	1513
Total	*2810*	*2987*	*2964*	*1254.2*	*1313.2*	*1312.6*	*2049.0*

Plot 1. Distribution of Reserves for Bayesian model



Appendix A.

Cumulative claims from different lines of business.

Triangle k = 1 **Development Year**

Origin Year

i / j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	118	487	1232	1266	1266	1397	1397	1397	1492	1492	1492
1	124	657	863	890	914	916	941	941	941	865	865
2	556	2204	3494	2998	2983	3018	2458	2458	2470	2470	2470
3	1646	2351	2492	2507	2612	2612	2608	1755	1755	1755	1755
4	317	886	890	890	950	990	990	990	990	990	990
5	242	919	1218	1224	1229	1249	1249	1249	1249	1249	1249
6	203	612	622	639	667	647	647	647	647	647	647
7	492	1405	1685	1668	1753	1742	1804	1804	1804	1804	1804
8	321	1149	1728	1863	1877	1877	1877	1877	1877	1877	1877
9	609	1109	1283	1294	1253	1255	1255	1255	1255	1255	1255
10	492	1627	1622	1672	1672	1672	1672	1672	1621	1621	1621
11	397	793	868	889	964	964	964	964	964	964	
12	523	1098	1475	1489	1489	1489	1489	1489	1489		
14	1786	2951	3370	3029	3211	3289	3325	3325			
14	241	465	536	596	652	652	652				
15	327	622	577	583	583	583					
16	275	520	529	529	541						
17	89	327	378	382							
18	295	301	396								
19	151	406									
20	315										

Triangle k = 2 **Development Year**

Origin Year

i / j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	92	442	541	541	528	528	528	528	528	528	528
1	451	1077	1085	1178	1212	1217	1217	1217	1217	1217	1217
2	404	717	834	849	849	850	850	850	850	850	850
3	203	572	813	875	878	910	912	1096	1089	1089	986
4	352	834	1048	1072	1088	1088	1088	1088	1088	1088	1088
5	504	1246	1272	1353	1285	1285	1285	1285	1285	1285	1285
6	509	1008	1061	1061	1061	1071	1071	1071	1071	1071	1071
7	229	580	630	670	672	672	672	672	672	672	672
8	324	815	871	859	867	777	777	777	777	777	777
9	508	805	906	969	971	971	971	971	971	971	971
10	354	641	833	842	842	842	842	842	842	842	842
11	431	847	854	915	918	918	918	918	918	918	
12	205	830	978	1034	1048	1048	1048	1048	1048		
14	522	1134	1064	1202	1202	1210	1210	1210			
14	567	925	915	957	953	953	953				
15	1238	1924	2034	1897	1897	1897					
16	355	1003	1137	1164	1196						
17	312	680	682	686							
18	246	352	418								
19	91	418									
20	130										

Triangle k = 3 **Development Year**

Origin Year

i / j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	268	456	485	483	483	483	483	483	483	483	483
1	268	520	577	579	579	579	579	579	579	579	579
2	385	968	1017	1019	1019	1019	1019	1019	1019	1019	1019
3	251	742	795	931	931	931	931	931	931	931	931
4	456	905	1162	1164	1164	1164	1164	1164	1191	1191	1191
5	477	1286	1376	1376	1373	1373	1373	1373	1373	1373	1373
6	405	999	1172	1196	1196	1210	1210	1210	1210	1210	1210
7	443	932	952	965	984	992	1012	1012	1012	1012	1012
8	477	1046	1336	1362	1375	1375	1375	1375	1375	1375	1375
9	581	1146	1316	1362	1391	1391	1391	1391	1391	1391	1391
10	401	997	1229	1248	1281	1284	1264	1264	1264	1264	1264
11	474	778	939	1321	1366	1392	1392	1392	1392	1392	
12	649	1420	1707	1709	1709	1709	1709	1638	1638		
14	911	1935	2304	2307	2309	2309	2309	2362			
14	508	1054	1101	1071	1071	1071	1071				
15	389	790	868	909	1569	1569					
16	373	998	1091	1155	1201						
17	276	853	932	948							
18	465	820	859								
19	343	622									
20	254										

Triangle k = 4 **Development Year**

Origin Year

i / j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	330	1022	1066	1086	1094	1094	1094	1094	1094	1094	1094
1	327	873	1057	1076	1082	1082	1082	1082	1082	1082	1082
2	304	1137	1234	1460	1475	1588	1586	1586	1586	1586	1586
3	426	1289	1418	1574	1578	1634	2250	2044	2044	2044	2044
4	750	2158	2910	3071	3213	3199	3052	3052	3052	3052	3052
5	761	2164	2446	2570	2578	2558	2558	2558	2558	2558	2558
6	1119	2666	2946	3008	3021	3022	3019	3019	3019	3019	3019
7	917	2458	2892	3502	3629	3664	3887	3867	3697	3697	3697
8	905	2014	2459	2466	2554	2554	2554	2540	2540	2540	2540
9	1761	2990	3235	3795	3816	3841	3842	3860	3860	3860	3860
10	824	2063	2378	2368	2384	2368	2373	2373	2373	2373	2373
11	4364	6630	6850	6885	6923	6923	6923	6923	6923	6923	
12	493	1587	1780	1794	1838	1838	1838	1865	1865		
14	4092	7710	6596	7201	7292	7292	7292	7292			
14	1733	3647	3699	3780	3773	3773	3733				
15	1261	2658	3063	3036	3093	3095					
16	1517	3054	3335	3438	3438						
17	778	1212	1247	1215							
18	727	1661	1816								
19	561	1486									
20	459										

Triangle k = 5 **Development Year**

Origin Year

i / j	0	1	2	3	4	5	6	7	8	9	10
0	486	964	1057	1106	1130	1130	1138	1131	1131	1131	1131
1	867	1669	1643	1717	1720	1724	1724	1724	1724	1724	1724
2	1285	1925	2204	2488	2507	2509	2510	2510	2436	2436	2436
3	395	994	1309	1442	1467	1467	1477	1477	1477	1477	1477
4	802	1468	1776	1823	1827	1832	1833	1833	1833	1833	1833
5	966	1967	2628	2743	2294	2338	2358	2358	2358	2358	2358
6	759	1766	1922	1863	1886	1886	1886	1886	1886	1886	1886
7	1136	2139	2219	1921	1931	1944	1947	1867	1867	1867	1867
8	1467	2243	2553	2598	2598	2598	2598	2598	2598	2598	2598
9	1309	2521	2660	2640	2639	2641	2659	2659	2659	2659	2659
10	877	2170	2341	2420	2516	2516	2431	2431	2431	2468	2468
11	1004	1963	2260	2226	2226	2215	2215	2059	2059	2059	
12	1351	2579	2736	2759	2760	2766	2688	2737	2737		
14	906	2341	2667	2655	2655	2650	2650	2824			
14	563	1450	1575	1603	1654	1654	1675				
15	417	1006	1034	1049	1049	1050					
16	322	836	1046	1123	1143						
17	1047	1656	1689	1779							
18	497	843	877								
19	1021	1237									
20	302										

Triangle k = 6 **Development Year**

Origin Year

i / j	0	1	2	3	4	5	6	7	8	9	10
0	18	64	64	64	64	64	64	64	64	64	64
1	20	73	103	153	155	155	155	155	155	155	155
2	20	70	318	328	328	328	328	328	328	328	328
3	88	133	133	133	133	133	133	133	133	133	133
4	3	180	214	214	215	215	215	215	215	215	215
5	11	79	80	82	81	81	81	81	81	81	81
6	17	66	105	172	172	172	188	188	188	188	199
7	73	216	218	218	218	218	218	218	218	218	218
8	48	213	253	386	400	400	317	304	304	304	304
9	98	153	153	158	158	158	158	158	158	158	158
10	38	529	557	632	639	639	639	639	639	639	639
11	42	140	141	141	141	141	141	141	141	141	
12	64	95	95	102	102	102	102	102	102		
14	57	144	169	178	178	178	178	178			
14	85	178	188	186	186	186	186				
15	212	341	357	371	371	371					
16	56	152	187	246	246						
17	25	44	103	178							
18	19	137	140								
19	25	45									
20	7										

Appendix B. (Conjugate Families)

Theorem 1 (Normal Case)

Let $\lambda^{U}_{i,j,k} \left| f^{Ind}_{j,k}, \upsilon^{2}_{i,j,k} \right. \sim N\left(f^{Ind}_{j,k}, \upsilon^{2}_{i,j,k}\right)$ and $f^{Ind}_{j,k} \left| f^{Coll}_{j}, \tau^{2}_{j,k} \right. \sim N\left(f^{Coll}_{j}, \tau^{2}_{j,k}\right)$. Then the posterior for $f^{Ind}_{j,k} \left| \lambda^{U}_{i,j,k} \right.$ is distributed Normal with mean $f^{(1)}_{i,j,k}$ and variance $\left(\tau^{(1)}_{i,j,k}\right)^{-2}$

Proof.

$$\pi\left(f^{Ind}_{j,k} \middle| \lambda^{U}_{i,j,k}\right) \propto f\left(f^{Ind}_{j,k} \middle| f^{Coll}_{j}, \tau^{2}_{j,k}\right) \times \pi\left(\lambda^{U}_{i,j,k} \middle| f^{Ind}_{j,k}, \upsilon^{2}_{i,j,k}\right)$$

$$\pi\left(f^{Ind}_{j,k} \middle| \lambda^{U}_{i,j,k}\right) \propto \left(2\pi\tau^{2}_{j,k}\right)^{-1/2} \exp\left[-\frac{1}{2\tau^{2}_{j,k}}\left(f^{Ind}_{j,k} - f^{Coll}_{j}\right)^{2}\right] \times \left(2\pi\upsilon^{2}_{i,j,k}\right)^{-1/2} \exp\left[-\frac{1}{2\upsilon^{2}_{i,j,k}}\left(\lambda^{U}_{i,j,k} - f^{Ind}_{j,k}\right)^{2}\right]$$

$$\propto \exp\left[-\frac{1}{2\tau^{2}_{j,k}}\left(f^{Ind}_{j,k} - f^{Coll}_{j}\right)^{2}\right] \times \exp\left[-\frac{1}{2\upsilon^{2}_{i,j,k}}\left(\lambda^{U}_{i,j,k} - f^{Ind}_{j,k}\right)^{2}\right]$$

$$= \exp\left[-\frac{1}{2\tau^{2}_{j,k}}\left(\left(f^{Ind}_{j,k}\right)^{2} - 2f^{Ind}_{j,k}f^{Coll}_{j} + \left(f^{Coll}_{j}\right)^{2}\right) - \frac{1}{2\upsilon^{2}_{i,j,k}}\left(\left(\lambda^{U}_{i,j,k}\right)^{2} - 2\lambda^{U}_{i,j,k}f^{Ind}_{j,k} + \left(f^{Ind}_{j,k}\right)^{2}\right)\right]$$

$$= \exp\left[-\frac{1}{2\tau^{2}_{j,k}\upsilon^{2}_{i,j,k}}\left(\upsilon^{2}_{i,j,k}\left(f^{Ind}_{j,k}\right)^{2} - 2\upsilon^{2}_{i,j,k}f^{Ind}_{j,k}f^{Coll}_{j} + \upsilon^{2}_{i,j,k}\left(f^{Coll}_{j}\right)^{2}\right) - \tau^{2}_{j,k}\left(\lambda^{U}_{i,j,k}\right)^{2} - 2\tau^{2}_{j,k}\lambda^{U}_{i,j,k}f^{Ind}_{j,k} + \tau^{2}_{j,k}\left(f^{Ind}_{j,k}\right)^{2}\right]$$

$$= \exp\left[-\frac{1}{2\tau^{2}_{j,k}\upsilon^{2}_{i,j,k}}\left(\left(f^{Ind}_{j,k}\right)^{2}\left(\upsilon^{2}_{i,j,k} + \tau^{2}_{j,k}\right) - 2f^{Ind}_{j,k}\left(\upsilon^{2}_{i,j,k}f^{Coll}_{j} + \tau^{2}_{j,k}\lambda^{U}_{i,j,k}\right) + \left(\upsilon^{2}_{i,j,k}f^{Coll}_{j} - \tau^{2}_{j,k}\lambda^{U}_{i,j,k}\right)\right)\right]$$

$$\propto \exp\left[-\frac{1}{2\tau^{2}_{j,k}\upsilon^{2}_{i,j,k}}\left(\left(f^{Ind}_{j,k}\right)^{2}\left(\upsilon^{2}_{i,j,k} + \tau^{2}_{j,k}\right) - 2f^{Ind}_{j,k}\left(\upsilon^{2}_{i,j,k}f^{Coll}_{j} + \tau^{2}_{j,k}\lambda^{U}_{i,j,k}\right)\right)\right]$$

$$= \exp\left[-\frac{1}{2}\left(\left(f^{Ind}_{j,k}\right)^{2}\left(\frac{1}{\tau^{2}_{j,k}} + \frac{1}{\upsilon^{2}_{i,j,k}}\right) - 2f^{Ind}_{j,k}\left(\frac{f^{Coll}_{j}}{\tau^{2}_{j,k}} + \frac{\lambda^{U}_{i,j,k}}{\upsilon^{2}_{i,j,k}}\right)\right)\right]$$

$$= \exp\left[-\frac{1}{2}\left(\frac{1}{\tau^{2}_{j,k}} + \frac{1}{\upsilon^{2}_{i,j,k}}\right)\left[\frac{\left(f^{Ind}_{j,k}\right)^{2}\left(\frac{1}{\tau^{2}_{j,k}} + \frac{1}{\upsilon^{2}_{i,j,k}}\right)}{\left(\frac{1}{\tau^{2}_{j,k}} + \frac{1}{\upsilon^{2}_{i,j,k}}\right)} - \frac{2f^{Ind}_{j,k}\left(\frac{f^{Coll}_{j}}{\tau^{2}_{j,k}} + \frac{\lambda^{U}_{i,j,k}}{\upsilon^{2}_{i,j,k}}\right)}{\left(\frac{1}{\tau^{2}_{j,k}} + \frac{1}{\upsilon^{2}_{i,j,k}}\right)}\right]\right]$$

$$= \exp\left[-\frac{1}{2}\left(\frac{1}{\tau_{j,k}^2}+\frac{1}{\upsilon_{i,j,k}^2}\right)\left[\left(f_{j,k}^{Ind}\right)^2 - \frac{2f_{j,k}^{Ind}\left(\frac{f_j^{Coll}}{\tau_{j,k}^2}+\frac{\lambda_{i,j,k}^U}{\upsilon_{i,j,k}^2}\right)}{\left(\frac{1}{\tau_{j,k}^2}+\frac{1}{\upsilon_{i,j,k}^2}\right)}\right] + w - w\right]$$

$$= \exp\left[-\frac{1}{2}\left(\frac{1}{\tau_{j,k}^2}+\frac{1}{\upsilon_{i,j,k}^2}\right)\left[\left(f_{j,k}^{Ind}\right)^2 - \frac{2f_{j,k}^{Ind}\left(\frac{f_j^{Coll}}{\tau_{j,k}^2}+\frac{\lambda_{i,j,k}^U}{\upsilon_{i,j,k}^2}\right)}{\left(\frac{1}{\tau_{j,k}^2}+\frac{1}{\upsilon_{i,j,k}^2}\right)}\right] + w\right] \times \exp[-w]$$

$$\propto \exp\left[-\frac{1}{2}\left(\frac{1}{\tau_{j,k}^2}+\frac{1}{\upsilon_{i,j,k}^2}\right)\left[\left(f_{j,k}^{Ind}\right)^2 - \frac{2f_{j,k}^{Ind}\left(\frac{f_j^{Coll}}{\tau_{j,k}^2}+\frac{\lambda_{i,j,k}^U}{\upsilon_{i,j,k}^2}\right)}{\left(\frac{1}{\tau_{j,k}^2}+\frac{1}{\upsilon_{i,j,k}^2}\right)}\right] + w\right]$$

Let $w = \left(\frac{\left(\frac{f_j^{Coll}}{\tau_{j,k}^2}+\frac{\lambda_{i,j,k}^U}{\upsilon_{i,j,k}^2}\right)}{\left(\frac{1}{\tau_{j,k}^2}+\frac{1}{\upsilon_{i,j,k}^2}\right)}\right)^2$

We obtain

$$\pi\left(f_{j,k}^{Ind} \middle| \lambda_{i,j,k}^U\right) \propto \exp\left[-\frac{1}{2}\left(\frac{1}{\tau_{j,k}^2}+\frac{1}{\upsilon_{i,j,k}^2}\right)\left[\left(f_{j,k}^{Ind}\right) - \frac{\left(\frac{f_j^{Coll}}{\tau_{j,k}^2}+\frac{\lambda_{i,j,k}^U}{\upsilon_{i,j,k}^2}\right)}{\left(\frac{1}{\tau_{j,k}^2}+\frac{1}{\upsilon_{i,j,k}^2}\right)}\right]^2\right]$$

$$= \exp\left[-\frac{1}{2}\left(\tau_{i,j,k}^{(1)}\right)^{-2}\left[\left(f_{j,k}^{Ind}\right) - f_{i,j,k}^{(1)}\right]^2\right]$$

with

$$f_{i,j,k}^{(1)} = \frac{\left(\frac{f_j^{Coll}}{\tau_{j,k}^2}+\frac{\lambda_{i,j,k}^U}{\upsilon_{i,j,k}^2}\right)}{\left(\frac{1}{\tau_{j,k}^2}+\frac{1}{\upsilon_{i,j,k}^2}\right)}$$

$$\left(\tau_{i,j,k}^{(1)}\right)^{-2} = \left(\frac{1}{\tau_{j,k}^2}+\frac{1}{\upsilon_{i,j,k}^2}\right)$$

Appendix C. (WinBUGS Code)

```
model {
# Enter data as an incremental claims triangle
        for(k in 1:K) {
                for(i in 1:I) {
                        z[i,1,k] <- y[i,1,k] }

 # Calculate the cumulative claims triangle
        for(j in 1:J-1) {
                for( i in 1:(I-j)) {
                         z[i,j+1,k] <- y[i,j+1,k] + z[i,j,k] } } }

# Define the development factors for the observed data
        for(k in 1:K) {
                for(j in 1:J-1) {
                        for( i in 1:(I-j)) {
        # Formula (3.3)
                                lambda[i,j,k] <-z[i,j+1,k]/ z[i,j,k] } }

# Define distributional assumptions on the development factors. Formula (5.1)
# First stage. Formula (5.2)
        for(j in 1:J-1) {
                for( i in 1:(I-j)) {
        # Formula (4.2)
                        lambda[i,j,k] ~ dnorm(f.ind[j,k], u.df[i,j,k])
        # Formula (4.5)
                        u.df[i,j,k] <- z[i,j,k]/s[j,k] } } }
# Second stage. Formula (5.4)
        for(k in 1:K) {
                for(j in 1:10) {
        # Formula (4.3)
                        f.ind[j,k] ~ dnorm(f.coll[j], tau[j,k])
        # Formula (5.5)
                        tau[j,k] <- 1.0E-6 }}

# Use log link for expected development factor so lower bound is zero
# Use vague independent normal priors on theta. Formula
        for(j in 1:(J-1)) {
        # Formula (5.7)
                log(f.coll[j]) <- theta[j]
        # Formula (5.6)
                theta[j] ~ dnorm(0,1.0E-6) }
```

```
# Define the stochastic model for future cumulative claims
# Using a Normal distribution is consistent with Mack's model as a GLM,
# although this could lead to negative cumulative amounts. Formula (5.18)
        for(k in 1:K) {
                for(i in 12:I) {
                        z.pred[i,I-i+1,k] <- z[i,I-i+1,k]
                        for(j in (I-i+1):(J-1)) {
                                z.pred[i,j+1,k] ~ dnorm(z.mu[i,j+1,k],z.tau[i,j+1,k])
                                z.mu[i,j+1,k] <- z.pred[i,j,k] * f.ind[j,k]
                                z.tau[i,j+1,k] <- max(1/(z.pred[i,j,k]*s[j,k]),1.0E-6)
                                y.pred[i,j+1,k]<- z.pred[i,j+1,k] - z.pred[i,j,k] } } }
# Formula (5.19) and (5.20)
        for(k in 1:K) {
                for(i in 12:I) {
                        outstand.row[i-10,k] <- sum(y.pred[i,(I+2-i):J,k]) }
                        outstand.row[I-9,k] <- sum(outstand.row[2:11,k]) }
                        Total <- sum(outstand.row[I-9,1:6]) }
# Initial Vaues
list(theta = c(0,0,0,0,0,0,0,0,0,0))


# Data. Load the values for K,I and J first, then load the s and y values
list(K=6, I = 21, J =11)


# Variance values s
s[,1]    s[,2]    s[,3]    s[,4]    s[,5]    s[,6]
418.840  176.150  58.600   317.920  134.690  912.980
87.390   11.250   6.560    38.220   14.640   50.360
6.980    2.650    9.480    12.790   6.340    8.730
1.530    0.380    28.070   0.610    4.980    0.030
1.020    0.710    0.040    0.720    0.060    0.001
7.070    0.001    0.050    17.280   0.400    1.250
18.990   2.660    0.320    1.430    2.050    0.030
0.660    0.001    0.050    0.560    0.160    0.001
0.540    0.001    0.001    0.001    0.050    0.001
0.001    0.870    0.001    0.001    0.001    0.060
END

# Add the information for each triangle from the Appendix A
# Example: Triangle for k = 0
y[,1,1]  y[,2,1]  y[,3,1]  y[,4,1]  y[,5,1]  y[,6,1]  y[,7,1]  y[,8,1]  y[,9,1]  y[,10,1] y[,11,1]
118      369      745      34       0        131      0        0        95       0        0
124      533      206      27       24       2        25       0        0        -76      0
556      1648     1290     -496     -15      35       -560     0        12       0        0
1646     705      141      15       105      0        -4       -853     0        0        0
317      569      4        0        60       40       0        0        0        0        0
242      677      299      6        5        20       0        0        0        0        0
203      409      10       17       28       -20      0        0        0        0        0
492      913      280      -17      85       -11      62       0        0        0        0
321      828      579      135      14       0        0        0        0        0        0
609      500      174      11       -41      2        0        0        0        0        0
492      1135     -5       50       0        0        0        0        -51      0        0
397      396      75       21       75       0        0        0        0        0        NA
523      575      377      14       0        0        0        0        0        NA       NA
1786     1165     419      -341     182      78       36       0        NA       NA       NA
241      224      71       60       56       0        0        NA       NA       NA       NA
327      295      -45      6        0        0        NA       NA       NA       NA       NA
275      245      9        0        12       NA       NA       NA       NA       NA       NA
89       238      51       4        NA       NA       NA       NA       NA       NA       NA
295      6        95       NA       NA       NA       NA       NA       NA       NA       NA
151      255      NA       NA       NA       NA       NA       NA       NA       NA       NA
315      NA       NA       NA       NA       NA       NA       NA       NA       NA       NA
END
```

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